SOME NEW INTEGRAL INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES OF ABSOLUTE VALUES ARE CONVEX AND CONCAVE

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ABSTRACT. In this paper, we prove some new inequalities for the functions whose derivatives' absolute values are convex and concave by dividing the interval $[a, b]$ to $n + 1$ equal even subintervals. We obtain some new results involving intermediate values of $|f'|$ in $[a, b]$ by using some classical inequalities like Hermite-Hadamard, Hölder and Power-Mean.

Keywords: convex functions, concave functions, Hermite-Hadamard inequality, Power-Mean inequality.

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1. INTRODUCTION

The function $f : [a, b] \to \mathbb{R}$, is said to be convex, if we have

$$f(tx + (1 - t) y) \leq tf(x) + (1 - t) f(y)$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$. Geometrically, this means that if P, Q and R are three distinct points on the graph of f with Q between P and R, then Q is on or below chord PR. A huge amount of the researchers are interested in this definition and there are several papers based on convexity. Several new results and collections of studies on integral inequalities have been given in [4], also in [11], new Hadamard type inequalities for convex functions have been proved. In [7], by using a new integral identity some Hadamard type inequalities have been obtained. In [12], integral inequalities of Hadamard type for product of convex functions have been established. In [1], some new classes of functions on co-ordinates have been defined and some Hadamard type integral inequalities have been obtained for these classes of functions. In [2], more general inequalities for product of convex functions have been established. In [3], some companion of generalization of Ostrowski type inequalities have been given that can be reduced to Hadamard type inequalities by special choices.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and let $a, b \in I$, with $a < b$. The following double inequality

$$f(\frac{a + b}{2}) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}$$

is known in the literature as Hadamard's inequality, given in [5]. In [6], a new variant of Hadamard's inequality has been given by using conformable fractional integral operators. In [8],

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some generalizations have been established for Hadamard type inequalities via conformable fractional integrals. In [9], new results for fractional integrals have been proved by Green function. In [10], new improvements of Hadamard type inequalities have been provided. Both inequalities hold in the reversed direction if f is concave.

In a recent paper [11], the following Theorems have been proved:

Theorem 1.1. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* $\overset{\circ}{I}$ *such that* $f' \in L[a, b]$, *where* $a, b \in I$ *with* $a < b$. *If* $|f'|$ *is convex on* $[a, b]$ *then the following inequality holds:*

$$\left| \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{3b+a}{4}\right)}{2} - \frac{1}{b-a} \int_a^b f(u)\, du \right| \tag{1}$$
$$\leq \left(\frac{b-a}{96}\right) \left[|f'(a)| + 4 \left| f'\left(\frac{3a+b}{4}\right)\right| + 2\left|f'\left(\frac{a+b}{2}\right)\right| + 4\left|f'\left(\frac{a+3b}{2}\right)\right| + |f'(b)| \right].$$

Theorem 1.2. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* $\overset{\circ}{I}$ *such that* $f' \in L[a, b]$, *where* $a, b \in I$ *with* $a < b$. *If* $|f'|^q$ *is convex on* $[a, b]$ *for some fixed* $q > 1$, *then the following inequality holds:*

$$\left| \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{3b+a}{4}\right)}{2} - \frac{1}{b-a} \int_a^b f(u)\, du \right| \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{2}\right)^{\frac{1}{q}} \left(\frac{b-a}{16}\right) \tag{2}$$
$$\times \left\{ \left(\left| f\left(\frac{3a+b}{4}\right)\right|^q + |f'(a)|^q \right)^{\frac{1}{q}} + \left(\left| f\left(\frac{a+b}{2}\right)\right|^q + \left| f'\left(\frac{3a+b}{4}\right)\right|^q \right)^{\frac{1}{q}} \right.$$
$$\left. \left(\left| f\left(\frac{a+3b}{4}\right)\right|^q + \left| f'\left(\frac{a+b}{2}\right)\right|^q \right)^{\frac{1}{q}} + \left(\left| f\left(\frac{a+3b}{4}\right)\right|^q + |f'(b)|^q \right)^{\frac{1}{q}} \right\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.3. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* $\overset{\circ}{I}$ *such that* $f' \in L[a, b]$, *where* $a, b \in I$ *with* $a < b$. *If* $|f'|^q$ *is convex on* $[a, b]$ *for some fixed* $q \geq 1$, *then the following inequality holds:*

$$\left| \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{3b+a}{4}\right)}{2} - \frac{1}{b-a} \int_a^b f(u)\, du \right| \leq \left(\frac{1}{2}\right) \left(\frac{1}{3}\right)^{\frac{1}{q}} \left(\frac{b-a}{16}\right) \tag{3}$$
$$\times \left\{ \left(|f'(a)|^q + 2 \left| f'\left(\frac{3a+b}{4}\right)\right|^q \right)^{\frac{1}{q}} + \left(\left| f\left(\frac{a+b}{2}\right)\right|^q + 2\left| f'\left(\frac{3a+b}{4}\right)\right|^q \right)^{\frac{1}{q}} \right.$$
$$\left. \left(\left| f'\left(\frac{a+b}{2}\right)\right|^q + 2\left| f'\left(\frac{a+3b}{4}\right)\right|^q \right)^{\frac{1}{q}} + \left(2\left| f\left(\frac{a+3b}{4}\right)\right|^q + |f'(b)|^q \right)^{\frac{1}{q}} \right\}.$$

Theorem 1.4. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* $\overset{\circ}{I}$ *such that* $f' \in L[a, b]$, *where* $a, b \in I$ *with* $a < b$. *If* $|f'|$ *is concave on* $[a, b]$ *for some fixed* $q > 1$, *then the following inequality*

holds:

$$\left| \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{3b+a}{4}\right)}{2} - \frac{1}{b-a} \int_a^b f(u)\, du \right| \tag{4}$$

$$\leq \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}} \left(\frac{b-a}{16}\right) \left[\left| f'\left(\frac{7a+b}{8}\right)\right| + \left| f'\left(\frac{5a+3b}{8}\right)\right| \right.$$

$$\left. + \left| f'\left(\frac{3a+5b}{8}\right)\right| + \left| f'\left(\frac{a+7b}{8}\right)\right| \right],$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.5. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* $\overset{\circ}{I}$ *such that* $f' \in L\left[a,b\right]$, *where* $a, b \in I$ *with* $a < b$. *If* $|f'|^q$ *is concave on* $[a,b]$ *for some fixed* $q > 1$, *then the following inequality holds:*

$$\left| \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{3b+a}{4}\right)}{2} - \frac{1}{b-a} \int_a^b f(u)\, du \right| \tag{5}$$

$$\leq \left(\frac{b-a}{32}\right) \left[\left| f'\left(\frac{13a+3b}{12}\right)\right| + \left| f'\left(\frac{11a+5b}{12}\right)\right| \right.$$

$$\left. + \left| f'\left(\frac{5a+13b}{12}\right)\right| + \left| f'\left(\frac{3a+13b}{12}\right)\right| \right].$$

The main aim of this paper is to establish some new inequalities involving values of $|f'|$ at intermediate points of $[a,b]$ interval for functions whose absolute values of derivatives are convex and concave. In order to prove our results we divide the interval $[a,b]$ to $n+1$ equal even sub-intervals.

2. Main results

We need following lemma to prove our main Theorems:

Lemma 2.1. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* $\overset{\circ}{I}$ *where* $a, b \in I$ *with* $a < b$. *If* $f' \in L\left[a,b\right]$ *and* n *is an odd number then the following equality holds:*

$$\left(\frac{b-a}{n+1}\right) \sum_{k=0}^{(n-1)/2} \left[\int_0^1 t f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1} + (1-t)\frac{a(n-2k+1)+b.2k}{n+1} \right) dt \right.$$

$$+ \int_0^1 (1-t).f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1} + (1-t)\frac{a(n-2k)+b.(2k+1)}{n+1} \right) dt \right] \tag{6}$$

$$= \sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1} \right) - \frac{n+1}{b-a} \int_a^b f(x)\, dx.$$

Proof. Firstly, we take

$$I_{1k} = \int_0^1 \left[t f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1} + (1-t)\frac{a(n-2k+1)+b.2k}{n+1} \right) \right] dt$$

$$I_{2k} = \int_0^1 \left[(1-t) f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1} + (1-t)\frac{a(n-2k)+b.(2k+1)}{n+1} \right) \right] dt.$$

Hence, it is obvious that for $k = 0$, we have

$$I_{10} + I_{20} = \int_0^1 t f'\left(t\frac{an+b}{n+1} + (1-t)a\right) dt + \int_0^1 t f'\left(t\frac{a(n-1)+2b}{n+1} + (1-t)\frac{an+b}{n+1}\right) dt.$$

Integrating by parts, we obtain

$$\int_0^1 t f'\left(t\frac{an+b}{n+1} + (1-t)a\right) dt + \int_0^1 (t-1) f'\left(t\frac{a(n-1)+2b}{n+1} + (1-t)\frac{an+b}{n+1}\right) dt$$

$$= \left[\frac{(n+1)t}{b-a} f\left(t\frac{an+b}{n+1} + (1-t)a\right)\Big|_0^1 - \frac{n+1}{b-a}\int_0^1 f\left(t\frac{an+b}{n+1} + (1-t)a\right) dt\right]$$

$$+ \left[\frac{(n+1)(t-1)}{b-a} f\left(t\frac{a(n-1)+2b}{n+1} + (1-t)\frac{an+b}{n+1}\right)\Big|_0^1\right.$$

$$\left. - \frac{n+1}{b-a}\int_0^1 f\left(t\frac{a(n-1)+2b}{n+1} + (1-t)\frac{an+b}{n+1}\right) dt\right].$$

By making use of the substitutions $x = t.\frac{an+b}{n+1} + (1-t)a$ and $y = t\frac{a(n-1)+2b}{n+1} + (1-t)\frac{an+b}{n+1}$, we get

$$I_{10} + I_{20} = \frac{2(n+1)}{b-a} f\left(\frac{an+b}{n+1}\right) - \left(\frac{n+1}{b-a}\right)^2 \left[\int_a^{\frac{an+b}{n+1}} f(x)\, dx + \int_{\frac{an+b}{n+1}}^{\frac{a(n-1)+2b}{n+1}} f(x) dx\right].$$

By the similar argument, for $k = 1$, we have

$$I_{11} + I_{21} = \frac{2(n+1)}{b-a} f\left(\frac{a(n-2)+3b}{n+1}\right) - \left(\frac{n+1}{b-a}\right)^2 \left[\int_{\frac{a(n-1)+2b}{n+1}}^{\frac{a(n-2)+3b}{n+1}} f(x)\, dx + \int_{\frac{a(n-2)+3b}{n+1}}^{\frac{a(n-3)+4b}{n+1}} f(x) dx\right].$$

If we apply same calculations from $k = 2$ to $(n-1)/2$, we have

$$\frac{b-a}{n+1} \sum_{k=0}^{(n-1)/2} (I_{1k} + I_{2k}) = \sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right) - \frac{n+1}{b-a}\int_a^b f(x)\, dx.$$

Which is the desired result. $\qquad\square$

Theorem 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\overset{\circ}{I}$ where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$ and n is an odd number then the following inequality holds:*

$$\left|\sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right) - \frac{n+1}{b-a}\int_a^b f(x)\, dx\right|$$

$$\leq \frac{b-a}{6(n+1)} \sum_{k=0}^{(n-1)/2} \left(4\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|\right.$$

$$\left. + \left|f'\left(\frac{a(n-2k+1)+b(2k)}{n+1}\right)\right| + \left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|\right). \tag{7}$$

Proof. By using Lemma 2 and properties of triangle inequality, we have

$$\left| \sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1} \right) - \frac{n+1}{b-a} \int_a^b f(x)\, dx \right|$$

$$\leq \frac{b-a}{n+1} \sum_{k=0}^{(n-1)/2} \left(\int_0^1 \left| tf'\left(t\frac{a(n-2k)+b(2k+1)}{n+1} + (1-t)\frac{a(n-2k+1)+b2k}{n+1} \right) \right| dt \right.$$

$$\left. + \int_0^1 \left| (1-t) f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1} + (1-t)\frac{a(n-2k)+b(2k+1)}{n+1} \right) \right| dt \right).$$

By using the convexity of $|f'|$, we obtain

$$\left| \sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1} \right) - \frac{n+1}{b-a} \int_a^b f(x)\, dx \right|$$

$$\leq \frac{b-a}{n+1} \sum_{k=0}^{(n-1)/2} \left| f'\left(\frac{a(n-2k)+b(2k+1)}{n+1} \right) \right| \int_0^1 t^2 dt + \left| f'\left(\frac{a(n-2k+1)+b.2k}{n+1} \right) \right| \int_0^1 t(1-t)dt$$

$$+ \left| f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1} \right) \right| \int_0^1 t(1-t)\, dt + \left| f'\left(\frac{a(n-2k)+b.(2k+1)}{n+1} \right) \right| \int_0^1 (1-t)^2 dt \right)$$

$$= \frac{b-a}{6(n+1)} \sum_{k=0}^{(n-1)/2} \left(4\left| f'\left(\frac{a(n-2k)+b(2k+1)}{n+1} \right) \right| + \left| f'\left(\frac{a(n-2k+1)+b.2k}{n+1} \right) \right| + \left| f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1} \right) \right| \right).$$

Which completes the proof.   □

Corollary 2.1. *If we choose $n = 1$ in (7) we obtain the following result:*

$$\left| f\left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \leq \frac{b-a}{6} \left(4\left| f'\left(\frac{a+b}{2} \right) \right| + |f'(a)| + |f'(b)| \right).$$

Corollary 2.2. *Under the conditions of Theorem 2.1, the following inequality holds:*

$$\left| \sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1} \right) - \frac{n+1}{b-a} \int_a^b f(x)\, dx \right|$$

$$\leq \frac{b-a}{n+1} \sum_{k=0}^{(n-1)/2} \left[\left(\frac{n-2k}{n+1} \right) |f'(a)| + \left(\frac{2k+1}{n+1} \right) |f'(b)| \right].$$

Besides if $|f'(x)| \leq M$, for all $x \in [a,b]$, then we have also the following inequality:

$$\left| \sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1} \right) - \frac{n+1}{b-a} \int_a^b f(x)\, dx \right|$$

$$\leq \left(\frac{b-a}{n+1} \right) \left(\frac{n-1}{2} \right) M.$$

Proof. It follows directly from Theorem 2.1 and using the convexity of $|f'|$.   □

Remark 2.1. *If we choose $n = 3$ in (7), this inequality reduces to (1).*

Theorem 2.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\overset{\circ}{I}$ where $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a, b]$ for some fixed $q > 1$ and n is an odd number, then the following inequality holds:*

$$\left| \sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right) - \frac{n+1}{b-a}\int_a^b f(x)\,dx \right|$$

$$\leq \left(\frac{b-a}{n+1}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \sum_{k=0}^{(n-1)/2} \left\{ \left[\frac{1}{2}\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q \right.\right.$$

$$\left. + \frac{1}{2}\left|f'\left(\frac{a(n-2k+1)+b(2k)}{n+1}\right)\right|^q\right]^{\frac{1}{q}} \tag{8}$$

$$+ \left[\frac{1}{2}\left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|^q + \frac{1}{2}\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q\right]^{\frac{1}{q}} \right\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2 and by using the Hölder inequality, we have

$$\left| \sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right) - \frac{n+1}{b-a}\int_a^b f(x)\,dx \right|$$

$$\leq \frac{b-a}{n+1} \sum_{k=0}^{(n-1)/2} \left\{ \left(\int_0^1 t^p dt\right)^{\frac{1}{p}} \right.$$

$$\times \left(\int_0^1 \left|f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1} + (1-t)\frac{a(n-2k+1)+b(2k)}{n+1}\right)\right|^q dt\right)^{\frac{1}{q}} \tag{9}$$

$$+ \left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}$$

$$\left. \times \left(\int_0^1 \left|f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1} + (1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q dt\right)^{\frac{1}{q}} \right\}$$

Since $|f|^q$ is convex on $[a, b]$ therefore we have

$$\int_0^1 \left|f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1} + (1-t)\frac{a(n-2k+1)+b(2k)}{n+1}\right)\right|^q dt$$

$$\leq \left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q \int_0^1 t\,dt + \left|f'\left(\frac{a(n-2k+1)+b(2k)}{n+1}\right)\right|^q \int_0^1 (1-t)\,dt$$

$$= \frac{1}{2}\left[\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q + \left|f'\left(\frac{a(n-2k+1)+b(2k)}{n+1}\right)\right|^q\right].$$

Similarly,

$$\left| \int_0^1 \left|f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1} + (1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q dt \right|$$

$$\leq \frac{1}{2}\left[\left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|^q + \left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q\right].$$

By using the last two inequalities in (9), we obtain the desired result. $\square$

Corollary 2.3. *If we choose $n = 1$ in (8), we obtain the following result:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le \left(\frac{b-a}{2}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^q$$

$$\times \left\{ \left[\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(a)|^q\right]^{\frac{1}{q}} + \left[|f'(b)|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}} \right\}.$$

Corollary 2.4. *Under the conditions of Theorem 2.2, the following inequality holds:*

$$\left| \sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right) - \frac{n+1}{b-a}\int_a^b f(x)\,dx \right|$$

$$\le \left(\frac{b-a}{n+1}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}} \sum_{k=0}^{(n-1)/2} \left\{ \left[\left(\frac{2n-4k+1}{n+1}\right)|f'(a)| + \left(\frac{4k+1}{n+1}\right)|f'(b)|\right]^{\frac{1}{q}} \right.$$

$$\left. + \left[\left(\frac{2n-4k-1}{n+1}\right)|f'(a)| + \left(\frac{4k+3}{n+1}\right)|f'(b)|\right]^{\frac{1}{q}} \right\}$$

Besides if $|f'(x)|^q \le M$, for all $x \in [a,b]$, then we have also the following inequality:

$$\left| \sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right) - \frac{n+1}{b-a}\int_a^b f(x)\,dx \right|$$

$$\le \left(\frac{b-a}{n+1}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{(n-1)^2}{2}\right)(M)^{\frac{1}{q}}.$$

Proof. It follows from Theorem 2.2 using the convexity of $|f'|^q$ and the fact
$\sum_{k=1}^n (u_k + v_k)^s \le \sum_{k=1}^n (u_k)^s + \sum_{k=1}^n (v_k)^s$, $u_k, v_k \ge 0$, $1 \le k \le n$, $0 \le s < 1$. $\square$

Remark 2.2. *If we choose $n = 3$ in (8), this inequality reduces to (2).*

Theorem 2.3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\overset{\circ}{I}$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q \ge 1$ and n is an odd number, then the following inequality holds:*

$$\left| \sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right) \right.$$

$$\left. - \frac{n+1}{b-a}\int_a^b f(x)\,dx \right|$$

$$\le \left(\frac{b-a}{n+1}\right)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^{\frac{1}{q}} \sum_{k=0}^{(n-1)/2} \left\{ \left[2\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q \right.\right. \tag{10}$$

$$\left. + \left|f'\left(\frac{a(n-2k+1)+b(2k)}{n+1}\right)\right|^q\right]^{\frac{1}{q}}$$

$$\left. + \left[2\left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|^q + \left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q\right]^{\frac{1}{q}} \right\}.$$

Proof. From Lemma 2 and by using the well-known power mean inequality, we have

$$
\left| \sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right) - \frac{n+1}{b-a} \int_a^b f(x)\,dx \right|
$$

$$
\leq \frac{b-a}{n+1} \sum_{k=0}^{(n-1)/2} \left\{ \left(\int_0^1 t\,dt\right)^{1-\frac{1}{q}}
\right.
$$

$$
\times \left(\int_0^1 t \left| f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1} + (1-t)\frac{a(n-2k+1)+b2k}{n+1}\right)\right|^q dt\right)^{\frac{1}{q}} \tag{11}
$$

$$
+ \left(\int_0^1 (1-t)\,dt\right)^{\left(1-\frac{1}{q}\right)}
$$

$$
\left.
\times \left(\int_0^1 (1-t)\left| f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1} + (1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q dt\right)^{\frac{1}{q}} \right\}.
$$

Since $|f|^q$ is convex on $[a,b]$ we have

$$
\int_0^1 t \left| f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1} + (1-t)\frac{a(n-2k+1)+b2k}{n+1}\right)\right|^q dt
$$

$$
\leq \left| f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q \int_0^1 t^2 dt + \left| f'\left(\frac{a(n-2k+1)+b2k}{n+1}\right)\right|^q \int_0^1 t(1-t)\,dt
$$

$$
= \frac{1}{3}\left| f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q + \frac{1}{6}\left| f'\left(\frac{a(n-2k+1)+b2k}{n+1}\right)\right|^q.
$$

Similarly,

$$
\left| \int_0^1 (1-t)\left| f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1} + (1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q dt \right|
$$

$$
\leq \frac{1}{6}\left| f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|^q + \frac{1}{3}\left| f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q.
$$

Using the last two inequalities in (11), we get the result. $\qquad\square$

Corollary 2.5. *If we choose $n = 1$ in (10), we obtain the following result:*

$$
\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \leq \left(\frac{b-a}{8}\right)\left(\frac{1}{3}\right)^{\frac{1}{q}}
$$

$$
\times \left[2\left| f'\left(\frac{a+b}{2}\right)\right|^q + |f'(a)|^q \right]^{\frac{1}{q}} + \left[2|f'(b)|^q + \left| f'\left(\frac{a+b}{2}\right)\right|^q \right]^{\frac{1}{q}}.
$$

Corollary 2.6. *Under the conditions of Theorem 2.3, using the same arguments as in Corollary 2.4, the following inequality holds:*

$$\left| \sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right) - \frac{n+1}{b-a}\int_a^b f(x)\,dx \right|$$

$$\leq \quad \left(\frac{b-a}{n+1}\right)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^{\frac{1}{q}} \sum_{k=0}^{(n-1)/2}\left\{\left[\left(\frac{3n-6k+1}{n+1}\right)|f'(a)|+\left(\frac{6k+2}{n+1}\right)|f'(b)|\right]^{\frac{1}{q}}\right.$$

$$\left. +\left[\left(\frac{3n-6k-2}{n+1}\right)|f'(a)|+\left(\frac{6k+5}{n+1}\right)|f'(b)|\right]^{\frac{1}{q}}\right..$$

Besides if $|f'(x)|^q \leq M$, for all $x \in [a,b]$, then we have also the following inequality:

$$\left| \sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right) - \frac{n+1}{b-a}\int_a^b f(x)\,dx \right|$$

$$\leq \quad \left(\frac{b-a}{n+1}\right)\left(\frac{1}{2}\right)^{\frac{1}{q}}(n-1)(M)^{\frac{1}{q}}.$$

Remark 2.3. *If we choose $n=3$ in (8), this inequality reduces to (3).*

Theorem 2.4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\overset{\circ}{I}$ such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a,b]$ for some fixed $q > 1$, and n is an odd number, then the following inequality holds:*

$$\left| \sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right) - \frac{n+1}{b-a}\int_a^b f(x)\,dx \right| \tag{12}$$

$$\leq \left(\frac{b-a}{n+1}\right)\left(\frac{q-1}{2q-1}\right)\sum_{k=0}^{(n-1)/2}\left(\left|f'\left(\frac{a(2n-4k+1)+b(4k+1)}{2(n+1)}\right)\right|^q + \left|f'\left(\frac{a(2n-4k-1)+b(4k+3)}{2(n+1)}\right)\right|^q\right)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2 and using the well-known Hölder inequality for $q > 1$ and $p = \frac{q}{q-1}$, we have

$$\left| \sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right) - \frac{n+1}{b-a}\int_a^b f(x)\,dx \right|$$

$$\leq \frac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}\left(\left(\int_0^1 t^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_0^1\left|f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b2k}{n+1}\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left. +\left(\int_0^1(1-t)^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\left(\int_0^1\left|f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q dt\right)^{\frac{1}{q}}\right.. \tag{13}$$

Since $|f|^q$ is concave on $[a,b]$ and by using the Hadamard inequality for concave functions, we have

$$\int_0^1\left|f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b2k}{n+1}\right)\right|^q dt$$

$$\leq \left|f'\left(\frac{\frac{a(n-2k)+b(2k+1)}{n+1}+\frac{a(n-2k+1)+b2k}{n+1}}{2}\right)\right|^q = \left|f'\left(\frac{a(2n-4k+1)+b(4k+1)}{2(n+1)}\right)\right|^q$$

and similarly,

$$\int\limits_0^1 \left| f' \left(t.\frac{a(n-2k-1)+b(2k+2)}{n+1} + (1-t)\frac{a(n-2k)+b.(2k+1)}{n+1} \right) \right|^q dt$$

$$\leq \left| f' \left(\frac{\frac{a(n-2k-1)+b(2k+2)}{n+1} + \frac{a(n-2k)+b.(2k+1)}{n+1}}{2} \right) \right|^q = \left| f' \left(\frac{a(2n-4k-1)+b(4k+3)}{2(n+1)} \right) \right|^q.$$

Using these two inequalities in (13), we get the desired result. $\qquad\square$

Corollary 2.7. *If we choose $n = 1$ in (12), we obtain the following result:*

$$\left| 2f\left(\frac{a+b}{2}\right) - \frac{2}{b-a}\int\limits_a^b f(x)\,dx \right|$$

$$\leq \left(\frac{b-a}{2}\right)\left(\frac{q-1}{2q-1}\right)\left[\left| f'\left(\frac{3a+b}{4}\right)\right|^q + \left| f'\left(\frac{3b+a}{4}\right)\right|^q \right].$$

Corollary 2.8. *Under the conditions of Theorem 2.4 and assume that $|f'|$ is a linear function, the following inequality holds:*

$$\left| \sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right) - \frac{n+1}{b-a}\int\limits_a^b f(x)\,dx \right|$$

$$\leq \left(\frac{b-a}{n+1}\right)\left(\frac{q-1}{2q-1}\right) 2 \sum_{k=0}^{(n-1)/2} \left[\left(\frac{n-k}{n+1}\right)|f'(a)| + \left(\frac{k+1}{n+1}\right)|f'(b)| \right].$$

Proof. It follows directly from Theorem 2.4 and linearity of $|f'|$. $\qquad\square$

Remark 2.4. *If we choose $n = 3$ in (12), this inequality reduces to (4).*

Theorem 2.5. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\overset{\circ}{I}$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a,b]$ for some fixed $q > 1$ and n is an odd number, then the following inequality holds:*

$$\left| \sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right) - \frac{n+1}{b-a}\int\limits_a^b f(x)\,dx \right|$$

$$\leq \frac{b-a}{(n+1)} \sum_{k=0}^{(n-1)/2} \left[\left| f'\left(\frac{2}{3}\frac{a(n-2k)+b(2k+1)}{n+1} + \frac{1}{3}\frac{a(n-2k+1)+b(2k)}{n+1}\right) \right| \right. \tag{14}$$

$$+ \left. \left| f'\left(\frac{1}{3}\frac{a(n-2k-1)+b(2k+2)}{n+1} + \frac{2}{3}\frac{a(n-2k)+b(2k+1)}{n+1}\right) \right| \right].$$

Proof. We know that if $|f'|^q$ is concave, then $|f'|$ is also concave (See: [11]). From Lemma 2 we have

$$\left| \sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right) - \frac{n+1}{b-a}\int\limits_a^b f(x)\,dx \right|$$

$$\leq \frac{b-a}{n+1} \sum_{k=0}^{(n-1)/2} \left[\int\limits_0^1 t \left| f'\left(t.\frac{a(n-2k)+b(2k+1)}{n+1} + (1-t)\frac{a(n-2k+1)+b(2k)}{n+1}\right) \right| dt \right.$$

$$+ \int\limits_0^1 (1-t).\left| f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1} + (1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right) \right| dt \right]$$

Since $|f'|$ is concave, by using Jensen Inequality, we obtain

$$
\leq \frac{b-a}{n+1} \sum_{k=0}^{(n-1)/2} \left[\left(\int_0^1 t\,dt \right) \left| f'\left(\frac{\int_0^1 t\left(t\frac{a(n-2k)+b(2k+1)}{n+1} + (1-t)\frac{a(n-2k+1)+b(2k)}{n+1} \right)dt}{\int_0^1 t\,dt} \right) \right| \right.
$$
$$
\left. + \left(\int_0^1 (1-t)\,dt \right) \left| f'\left(\frac{\int_0^1 (1-t)\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1} + (1-t)\frac{a(n-2k)+b(2k+1)}{n+1} \right)dt}{\int_0^1 (1-t)\,dt} \right) \right| \right].
$$

which is equivalent to (14). This completes proof.   □

Corollary 2.9. *If we choose $n = 1$ in (14), we obtain the following result:*

$$
\left| 2f\left(\frac{a+b}{2} \right) - \frac{2}{b-a} \int_a^b f(x)\,dx \right|
$$
$$
\leq \frac{b-a}{4} \left[\left| f'\left(\frac{2a+b}{3} \right) \right| + \left| f'\left(\frac{a+2b}{3} \right) \right| \right].
$$

Corollary 2.10. *Under the conditions of Theorem 2.5 and assume that $|f'|$ is a linear function, the following inequality holds:*

$$
\left| \sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1} \right) - \frac{n+1}{b-a} \int_a^b f(x)\,dx \right|
$$
$$
\leq \frac{2(b-a)}{n+1} \sum_{k=0}^{(n-1)/2} \left[(n-2k)|f'(a)| + (2k+n)|f'(b)| \right].
$$

Proof. It follows directly from Theorem 2.5 and linearity of $|f'|$.   □

Remark 2.5. *If we choose $n = 3$ in (14), this inequality reduces to (5).*

Remark 2.6. *We have obtained the generalization of Lemma 1 of $[11]$ and proved some more general inequalities that reduced to Theorem 1-5 of $[11]$ by selecting $n = 3$.*

3. Applications to the special means

We shall consider the means for arbitrary real number α and β $(\alpha \neq \beta)$. We take

$$
\begin{array}{ll}
H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}, \alpha, \beta \in \mathbb{R}^+ & \text{(Harmonic Mean)} \\
A(\alpha, \beta) = \frac{\alpha+\beta}{2}, \alpha, \beta \in \mathbb{R}^+ & \text{(Arithmetic Mean)} \\
G(\alpha, \beta) = \sqrt{\alpha\beta}, \alpha, \beta \in \mathbb{R}^+ & \text{(Geometric Mean)} \\
L(\alpha, \beta) = \frac{\alpha-\beta}{\ln\alpha-\ln\beta}, \alpha, \beta \neq 0, \alpha \neq \beta, \alpha, \beta \in \mathbb{R}^+ & \text{(Logaritmic Mean)}
\end{array}
$$

Proposition 3.1. *Under the conditions of Theorem 2.1 for n=1, if we choose $f(x) = x^2$ we have*

$$
\frac{1}{3}\left| G^2(a,b) - A^2(a,b) \right| \leq 2(b-a)A(a,b).
$$

Proposition 3.2. *Under the conditions of Theorem 2.1 for n=1, if we choose $f(x) = \frac{1}{x}$ we have*

$$
\left| A^{-1}(a,b) - L^{-1}(a,b) \right| \leq \frac{(b-a)}{6}\left(4A^{-2}(a,b) - \frac{1}{2}H^{-1}(a^2, b^2) \right).
$$

Proposition 3.3. *Under the conditions of Corollary 2.3, if we choose* $f(x) = x^2$ *we have*

$$\frac{1}{3}\left|G^2\left(a,b\right) - A^2\left(a,b\right)\right| \leq \frac{(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^q$$
$$\times\left\{\left[A^q\left(a,b\right) + a^q\right]^{\frac{1}{q}} + \left[A^q\left(a,b\right) + b^q\right]^{\frac{1}{q}}\right\}.$$

4. Conclusion

In the present study, a new integral identity has been established by dividing the interval $[a,b]$ to $n+1$ equal even subintervals. Then, several new integral inequalities are proved for convex and concave functions whose certain powers of derivatives absolute values of via this identity. The results found are a generalization that reveals many new inequalities as well as providing previous results. Some applications have also been given for our findings.

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